Functional renormalization group for ultracold fermions

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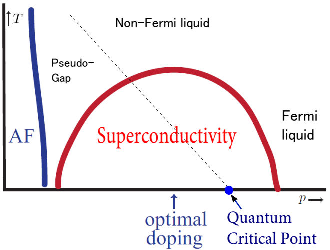
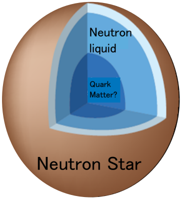
Introduction

Examples many-body fermionic systems

Many-body fermionic systems with nontrivial phases:

- Many-electron system: metal, insulators, magnetism, ...
- Nucleons: nuclear, nucleon superfluid inside neutron stars, ...
- Quarks in the high-density QCD





Effective field approach to strongly-correlated fermions

Microscopic model (Hubbard model, lattice spin model, lattice gauge theory)

$\downarrow$

Effective field theory

$\downarrow$

Experiments & Phenomenology

Effective field approach to strongly-correlated fermions

Microscopic model (Hubbard model, lattice spin model, lattice gauge theory)

$\downarrow$

Effective field theory

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Experiments & Phenomenology

Requirements for EFTs:

1. Be simpler than original microscopic models
2. Emerge from renormalizable theories, or lattice models.

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Experiments & Phenomenology

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Phenomenon	Effective Field Theory	Microscopic Model
Superconductivity	Ginzburg-Landau theory	BCS theory
Antiferromagnetism	Nonlinear sigma model	Heisenberg model
χ-symmetry breaking	NJL/QM model	QCD

Effective field approach to strongly-correlated fermions

Simple forms of effective action:

$$\mathcal{L} = \overline{\psi} G^{-1}(\partial_\tau, \nabla)\psi + g(\overline{\psi}\psi)^2$$

or

$$\mathcal{L} = \overline{\psi} G^{-1}(\partial_\tau, \nabla)\psi + \phi G_\phi^{-1}(\partial_\tau, \nabla)\phi + g_{\phi\psi}\phi\overline{\psi}\psi$$

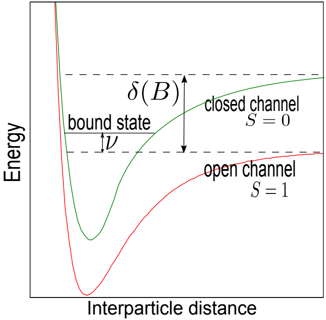
At low energies, interactions become strong due to dynamical effects.
$\Rightarrow$ Nonperturbative methods of QFT

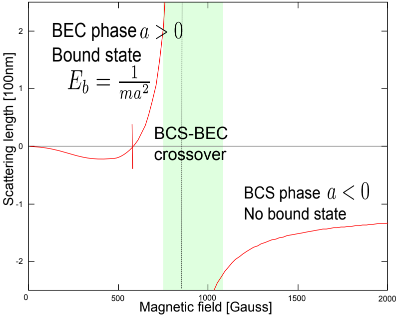
Important!

Nonperturbative techniques of field theories must be developed in order to describe IR physics using EFT.

Cold atomic physics

Ultracold fermions provides examples of strongly-correlated fermions.
High controllability can tune effective couplings with real experiments!



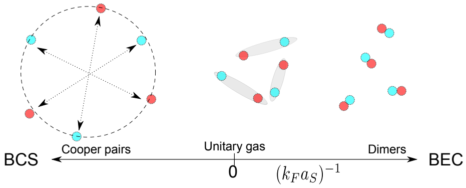


(Typically, $T \sim 100$nK, and $n \sim 10^{11-14}$ cm^{-3})

BCS-BEC crossover

EFT: Two-component fermions with an attractive contact interaction.

$$S=\int \mathrm{d}^4x\left[\overline{\psi}(x)\left(\partial_\tau-\frac{\nabla^2}{2m}-\mu\right)\psi(x)+g\overline{\psi}_1(x)\overline{\psi}_2(x)\psi_2(x)\psi_1(x)\right]$$



(Eagles 1969, Legget 1980, Nozieres & Schmitt-Rink 1985)

Question

Is it possible to treat EFT systematically to describe the BCS-BEC crossover?

Purpose of this talk

- Develop the functional renormalization group (FRG) method for many-body fermions.
- Study the BCS-BEC crossover using the developed formalism of FRG.
  - BCS side: Connection of FRG & BCS theory + GMB correction is made clear. Systematic improvement is considered to go beyond it!
  - BEC side: Describe the Bose gas of dimers /wo auxiliary field methods. This requires a new non-perturbative formalism of FRG.
  - Describe the whole region of the BCS-BEC crossover in this formalism.

Functional renormalization group

General framework of FRG

Generating functional of connected Green functions:

$$\exp(W[J]) = \int \mathcal{D}\Phi \exp\left(-S[\Phi] + J \cdot \Phi\right).$$

infinite dimensional integration!

Possible remedy: Construct nonperturbative relations of Green functions!
($\Rightarrow$ Functional techniques)

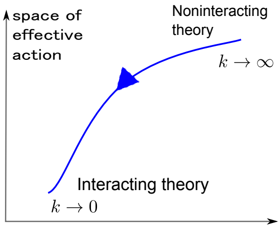
- Dyson-Schwinger equations
- 2PI formalism
- Functional renormalization group (FRG)

Flow equation of FRG

$\delta S_k[\Phi]$: Some function of Φ with a parameter k. (IR regulator)

k-dependent Schwinger functional

$$\exp(W_k[J]) = \int \mathcal{D}\Phi \exp\left[-\left(S[\Phi] + \delta S_k[\Phi]\right) + J \cdot \Phi\right]$$



Flow equation

$$\begin{aligned} -\partial_k W_k[J] &= \langle \partial_k \delta S_k[\Phi] \rangle_J \\ &= \exp\left(-W_k[J]\right) \partial_k(\delta S_k) \left[\delta/\delta J\right] \exp(W_k[J]) \end{aligned}$$

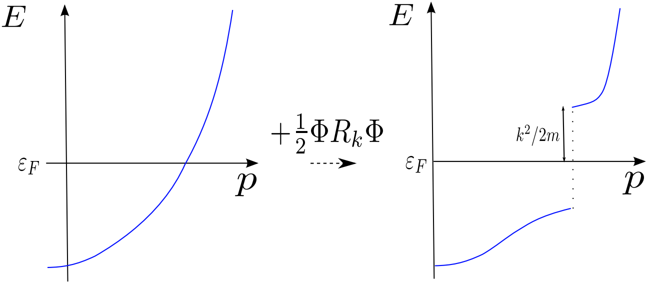
Consequence

We get a (functional) differential equation instead of a (functional) integration!

Conventional approach: Wetterich equation

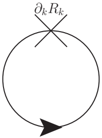
At high energies, perturbation theory often works well.
$\Rightarrow$ Original fields control physical degrees of freedom.

IR regulator for bare propagators ($\sim$ mass term): $\delta S_k[\Phi] = \frac{1}{2}\Phi_\alpha R_k^{\alpha\beta}\Phi_\beta$.



Flow equation of 1PI effective action $\Gamma_k[\Phi]$ (Wetterich 1993)

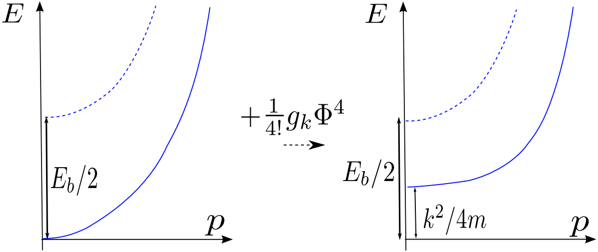
$$\partial_k \Gamma_k[\Phi] = \frac{1}{2}\mathrm{STr}\frac{\partial_k R_k}{\delta^2\Gamma_k[\Phi]/\delta\Phi\delta\Phi + R_k} =$$



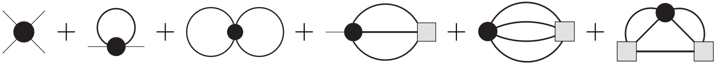
FRG beyond the naive one: vertex IR regulator

In the infrared region, collective bosonic excitations emerge quite in common.
(e.g.) Another low-energy excitation emerges in the $\Phi\Phi$ channel

Vertex IR regulator: $\delta S_k = \frac{1}{4!} g_k^{\alpha\beta\gamma\delta} \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta$.



Flow equation with the vertex IR regulator (YT, PTEP**2014**, 023A04)

$\partial_k \Gamma_k[\Phi] =$ 

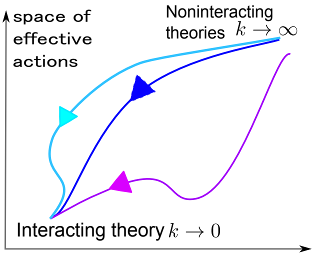
Optimization

Choice of IR regulators δS_k is arbitrary.

Optimization:
Choose the "best" IR regulator, which validates systematic truncation of an approximation scheme.

Optimization criterion (Litim 2000, Pawlowski 2007):

- IR regulators δS_k make the system gapped by a typical energy $k^2/2m$ of the parameter k.
- High-energy excitations ($\gtrsim k^2/2m$) should decouple from the flow of FRG at the scale k.
- Choose δS_k stabilizing calculations and making it easier.

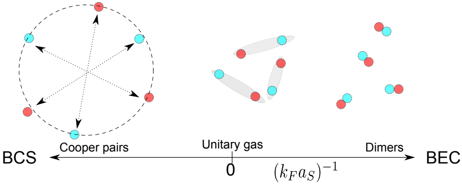


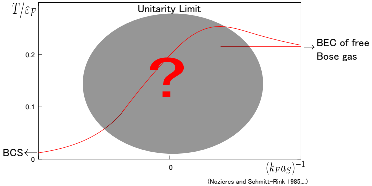
Application of fermionic FRG to the BCS-BEC crossover

BCS-BEC crossover

Model:

$$S=\int \mathrm{d}^4x\left[\overline{\psi}(x)\left(\partial_\tau-\frac{\nabla^2}{2m}-\mu\right)\psi(x)+g\overline{\psi}_1(x)\overline{\psi}_2(x)\psi_2(x)\psi_1(x)\right]$$





$(n=k_F^3/3\pi^2,\ \varepsilon_F=k_F^2/2m)$

Purpose of this talk

Nonperturbative FRG can describe the BCS-BEC crossover /wo auxiliary fields!

General strategy

We will calculate T_c/ε_F and μ/ε_F.
$\Rightarrow$ Critical temperature and the number density must be calculated.

We expand the 1PI effective action in the symmetric phase:

$$\begin{aligned}\Gamma_k[\overline{\psi},\psi] &= \beta F_k(\beta,\mu)+\int_p \overline{\psi}_p[G^{-1}(p)-\Sigma_k(p)]\psi_p \\ &\quad +\int_{p,q,q'}\Gamma^{(4)}_k(p)\overline{\psi}_{\uparrow,\frac{p}{2}+q}\overline{\psi}_{\downarrow,\frac{p}{2}-q}\psi_{\downarrow,\frac{p}{2}-q'}\psi_{\uparrow,\frac{p}{2}+q'}.\end{aligned}$$

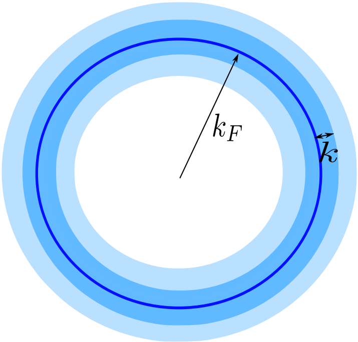
Critical temperature and the number density are determined by

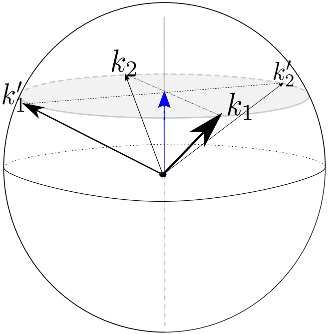
$$\frac{1}{\Gamma^{(4)}_0(p=0)}=0,\qquad n=\int_p\frac{-2}{G^{-1}(p)-\Sigma_0(p)}.$$

BCS side

Case 1 Negative scattering length $(k_F a_s)^{-1} \ll -1$.
$\Rightarrow$ Fermi surface exists, and low-energy excitations are fermionic quasi-particles.

Shanker's RG for Fermi liquid (Shanker 1994)



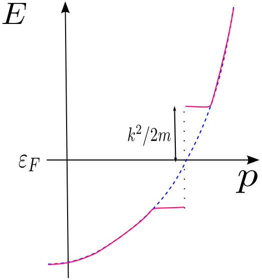


Functional implementation of Shanker's RG

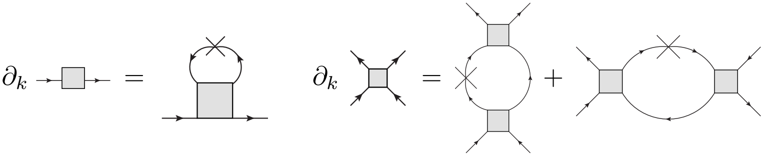
RG must keep low-energy fermionic excitations under control.

$\Rightarrow \delta S_k = \int_p \overline{\psi}_p R_k^{(f)}(\boldsymbol{p})\psi_p$ with

$$R_k^{(f)}(\boldsymbol{p}) = \mathrm{sgn}(\xi(\boldsymbol{p}))\left(\frac{k^2}{2m} - |\xi(\boldsymbol{p})|\right)\theta\left(\frac{k^2}{2m} - |\xi(\boldsymbol{p})|\right)$$

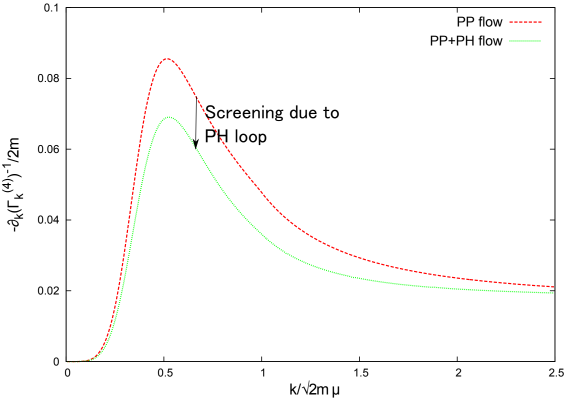


Flow equation of the self-energy Σ_k and the four-point 1PI vertex $\Gamma_k^{(4)}$:



Flow of fermionic FRG: effective four-fermion interaction

- Particle-particle loop $\Rightarrow$ RPA & BCS theory
- Particle-hole loop gives screening of the effective coupling at $k \sim k_F$

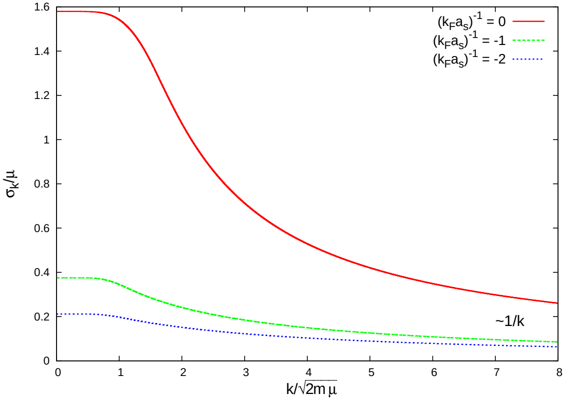


(YT, G. Fejős, T. Hatsuda, arXiv:1310.5800)

$$T_c^{\rm BCS} = \varepsilon_F \frac{8e^{\gamma_E-2}}{\pi} e^{-\pi/2k_F|a_s|} \Rightarrow T_c^{\rm BCS}/2.2.$$ (Gorkov, Melik-Barkhudarov, 1961)

Flow of fermionic FRG: self-energy

Local approximation on self-energy: $\Sigma_k(p) \simeq \sigma_k$.

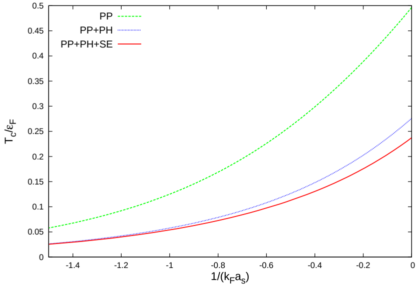


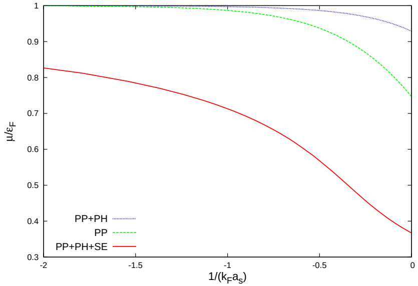
(YT, G. Fejős, T. Hatsuda, arXiv:1310.5800)

- High energy: $\sigma_k \simeq$ (effective coupling)$\times$(number density) $\sim 1/k$
- Low energy: $\partial_k \sigma_k \sim 0$ due to the particle-hole symmetry.

Transition temperature and chemical potential in the BCS side

(YT, G. Fejős, T. Hatsuda, arXiv:1310.5800)





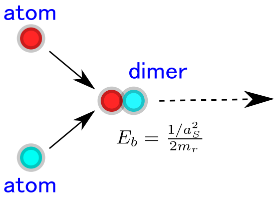
Consequence

- *Critical temperature* T_c/ε_F *is significantly reduced by a factor* 2.2 *in* $(k_F a_s)^{-1} \lesssim -1$, *and the self-energy effect on it is small in this region.*
- $\mu(T_c)/\varepsilon_F$ *is largely changed from* 1 *even when* $(k_F a_s)^{-1} \lesssim -1$.

BEC side

Case 2 Positive scattering length : $(k_F a_s)^{-1} \gg 1$

$\Rightarrow$ Low-energy excitations are one-particle excitations of composite dimers.



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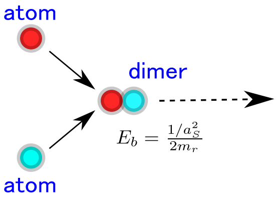


Several approaches for describing BEC of composite bosons. (Pros/Cons)

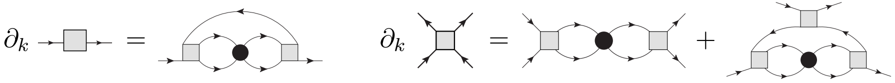
- Auxiliary field method
  (Easy treatment within MFA/ Fierz ambiguity in their introduction)
- Fermionic FRG ($\Leftarrow$ We develop this method!)
  (Unbiased and unambiguous/ Nonperturbative treatment is necessary)

Vertex IR regulator & Flow equation

Optimization can be satisfied with the vertex IR regulator:

$$\delta S_k = \int_p \frac{g^2 R_k^{(b)}(\boldsymbol{p})}{1 - g R_k^{(b)}(\boldsymbol{p})} \int_{q,q'} \overline{\psi}_{\uparrow, \frac{p}{2}+q} \overline{\psi}_{\downarrow, \frac{p}{2}-q} \psi_{\downarrow, \frac{p}{2}-q'} \psi_{\uparrow, \frac{p}{2}+q'}$$

Flow equation up to fourth order (YT, PTEP**2014** 023A04, YT, arXiv:1402.0283):



Effective boson propagator in the four-point function:

$$\frac{1}{\Gamma_k^{(4)}(p)} = -\frac{m^2 a_s}{8\pi} \left(i p^0 + \frac{\boldsymbol{p}^2}{4m} \right) - R_k^{(b)}(\boldsymbol{p})$$

Flow of fermionic FRG: self-energy

Flow equation of the self-energy:

$$\partial_k \Sigma_k(p) = \int_l \frac{\partial_k \Gamma_k^{(4)}(p+l)}{il^0 + \boldsymbol{l}^2/2m + 1/2ma_s^2 - \Sigma_k(l)}.$$

If $|\Sigma_k(p)| \ll 1/2ma_s^2$,

$$\begin{aligned}\Sigma_k(p) &\simeq \int_l \frac{\Gamma_k^{(4)}(p+l)}{il^0 + \boldsymbol{l}^2/2m + 1/2ma_s^2} \\ &\simeq \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{(8\pi/m^2 a_s) n_B(\boldsymbol{q}^2/4m + \frac{m^2 a_s}{8\pi} R_k^{(b)}(\boldsymbol{q}))}{ip^0 + \frac{\boldsymbol{q}^2}{4m} + \frac{m^2 a_s}{8\pi} R_k^{(b)}(\boldsymbol{q}) - \frac{(\boldsymbol{q}+\boldsymbol{p})^2}{2m} - \frac{1}{2ma_s^2}}.\end{aligned}$$

Estimate of $|\Sigma_k(p)|$:

$$|\Sigma_k(p)| \lesssim \frac{1}{2ma_s^2} \times (\sqrt{2mT}a_s)^3 \times n_B(k^2/4m).$$

$\Rightarrow$ Our approximation is valid up to $(k^2/2m)/T \sim (k_F a_s)^3 \ll 1$.

Critical temperature in the BEC side

Number density:

$$
\begin{aligned}
n &= \int_p \frac{-2}{ip^0 + \boldsymbol{p}^2/2m + 1/2ma_s^2 - \Sigma_0(p)} \\
&\simeq \frac{(2mT_c)^{3/2}}{\pi^2}\sqrt{\frac{\pi}{2}}\zeta(3/2).
\end{aligned}
$$

Critical temperature and chemical potential:

$$
T_c/\varepsilon_F = 0.218, \qquad \mu/\varepsilon_F = -1/(k_F a_s)^2.
$$

$\Rightarrow$ Transition temperature of BEC.

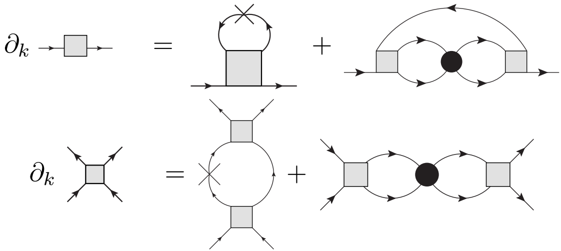
Consequence

FRG with vertex regulator provides a nonperturbative description of many-body composite particles.

fermionic FRG for the BCS-BEC crossover

We discuss the whole region of the BCS-BEC crossover with fermionic FRG.
$\Rightarrow$ Combine two different formalisms appropriate for BCS and BEC sides.

Minimal set of the flow equation for Σ_k and $\Gamma^{(4)}_k$:(YT, arXiv:1402.0283)



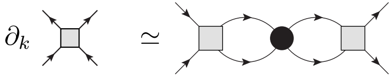
Flow of fermionic FRG with multiple regulators

Flow of four-point vertex:

Important property: fermions decouple from RG flow at the low energy region.

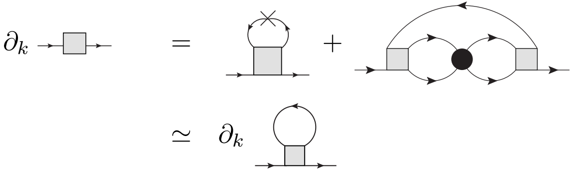
- In BCS side, fermions decouples due to Matsubara freq. ($k^2/2m \lesssim \pi T$).
- In BEC side, fermions decouples due to binding E. ($k^2/2m \lesssim 1/2ma_s^2$).

Approximation on the flow of the four-point vertex at low energy:



Flow of self-energy:

At a low-energy region, the above approx. gives



Qualitative behaviors of the BCS-BEC crossover from f-FRG

Approximations on the flow equation have physical interpretations.

Four-point vertex: Particle-particle RPA. The Thouless criterion $1/\Gamma^{(4)}(p=0)=0$ gives

$$\frac{1}{a_s}=-\frac{2}{\pi}\int_0^\infty\sqrt{2m\varepsilon}\mathrm{d}\varepsilon\left[\frac{\tanh\frac{\beta}{2}(\varepsilon-\mu)}{2(\varepsilon-\mu)}-\frac{1}{2\varepsilon}\right]$$

$\Rightarrow$ BCS gap equation at $T=T_c$.

Number density: $n=-2\int 1/(G^{-1}-\Sigma)$.

$$n=-2\int_p^{(T)}G(p)-\frac{\partial}{\partial\mu}\int_p^{(T)}\ln\left[1+\frac{4\pi a_s}{m}\left(\Pi(p)-\frac{m\Lambda}{2\pi^2}\right)\right].$$

$\Rightarrow$ Pairing fluctuations are taken into account. (Nozieres, Schmitt-Rink, 1985)

Consequence

We established the fermionic FRG which describes the BCS-BEC crossover.

Summary & Outlook

Summary

- EFT is a powerful approach to strongly-correlated fermions.
  $\Rightarrow$ More powerful analytical method is still required for intuitive, unbiased and systematic understandings.
- Fermionic FRG is a promising formalism.
  $\Rightarrow$ Separation of energy scales can be realized by optimization.
  $\Rightarrow$ Very flexible form for various approximation schemes.
- Fermionic FRG is applied to the BCS-BEC crossover.
  $\Rightarrow$ BCS side: GMB correction + the shift of Fermi energy from μ.
  $\Rightarrow$ BEC side: BEC without explicit bosonic fields.
  $\Rightarrow$ whole region: Crossover physics is successfully described at the quantitative level with a minimal setup on f-FRG.

Outlook

- Perform numerical computations for the whole region of the BCS-BEC crossover.
  $\Rightarrow$ This explicitly confirms that our formalism can be systematically improvable to describe the crossover physics.
- Application of fermionic FRG to other low-density strongly-correlated fermions.
  e.g., Neutron superfluid, dipolar fermions in ultracold atoms, ...